9/11 – Théorie de Hodge pour les variétés riemanniennes non compactes$^{+\varepsilon}$ (R. Pengo)

[GdT "Homologie stable réelle des groupes arithmétiques", ENS Lyon]

1. Rappels
2. Formules de Stokes
3. Formules d'adjonction
4. Andreotti–Vesentini
5. Prop. 2.5 de [Borel]
6. Le cadre pour le futur.

① M variété riemannienne, $\langle\cdot,\cdot\rangle_x : \mathrm{Sym}^2(T_x(M)) \to \mathbb{R}$, $x \in M$

- **Gradient** $\nabla : C^\infty(M,\mathbb{R}) \to C^\infty(M,TM)$, $|X(f)(x)|_x = \langle (\nabla f)(x), X(x)\rangle_x$

- **Distance** $d : M \times M \to \mathbb{R}$. Si M est **complète** alors:
  $\forall r>0\ \exists\, C_r \subseteq D_r \subseteq M$ compacts, $\sigma_r : M \to [0,1]$ lisses t.q.
  $C_r^\circ \subseteq C_{r'}$, $\forall r<r'$; $M = \bigcup_{r>0} C_r$; $\sigma_r(C_r)=1$; $\sigma_r(M\setminus D_r)=0$;

  $$|(\nabla\sigma_r)(x)|_x \leq \frac{c}{r},$$ où c dépend seulement de M.

- **Théorème de Hodge** $* : \Omega^k(M) \to \Omega^{d-k}(M)$, M complète et orientée,
  $\partial : \Omega^k(M) \to \Omega^{k-1}(M)$, $\Delta : \Omega^k(M) \to \Omega^k(M)$, $\Delta := d\circ\partial + \partial\circ d$
  $\omega := *(1)$, $\int_M f := \underbrace{\int_M f\omega}_{\in\, \mathbb{R}\cup\{+\infty\}}$, $\langle\cdot,\cdot\rangle_M : \mathrm{Sym}^2(\Omega^k(M)) \to \mathbb{R}$

$$\Omega^k_{(2)}(M) := \{\alpha \in \Omega^k(M) \mid \|\alpha\|_M := \sqrt{\langle \alpha, \alpha \rangle_M} < +\infty\}$$

$$\mathcal{H}^k(M) := \Omega^k(M)^{\Delta = 0}, \quad \mathcal{H}^k_{(2)}(M) := \mathcal{H}^k(M) \cap \Omega^k_{(2)}(M)$$

$$\underbrace{= \mathcal{H}^k_{(2)}(M)}$$

$$\begin{tikzcd}
\mathcal{H}^k_{(2)}(M)^{d=0} \arrow[r, hook] \arrow[dr, "\mu"'] & \Omega^k_{(2)}(M)^{d=0} \arrow[r, hook] \arrow[d] & \Omega^k(M)^{d=0} \arrow[d] \\
& H^k_{(2)}(M) \arrow[r, hook, "r"'] & H^k_{dR}(M)
\end{tikzcd}$$

Kodaira: μ surjective

Andreotti Vesentini: $\mathcal{H}^k_{(2)}(M) = \Omega^k_{(2)}(M)^{d = \delta = 0}$

Prop. 2.5 Borel: $\mathcal{H}^k_{(2)}(M) \cap d(\Omega^{k-1}_{(2)}(M)) = 0$

② (§1 de [Borel])

<u>Prop.</u>[1.3] M variété riemannienne complète, orientée. $X : M \to TM$, $f \in C^1(M, E)$ où E euclidien, telles que :

$$\mathcal{L}_X(\omega) = 0, \ \max_{x \in M} |X(x)|_x < +\infty, \ f, X(f) \in L^1(M, E).$$

$$\Rightarrow \int_M X(f) \cdot \omega = 0$$

Formule "magique" de Cartan

<u>Dém</u> $\mathcal{L}_X(\omega) := \frac{d}{dt}\Big|_{t=0} \left(\exp(X.t)^*(\omega) \right) \overset{\}}{=} d(i_X(\omega)) + i_X(d(\omega))$

$i_X : \Omega^k(M) \to \Omega^{k-1}(M)$, $i_X(\omega)(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1})$.

$$\int_M X(f)\cdot\omega$$

$X(f)\cdot\omega$ est exacte

$$\mathcal{L}_X(f\cdot\omega) = \underbrace{\mathcal{L}_X(f)}_{=X(f)}\cdot\omega + f\cdot\overbrace{\mathcal{L}_X(\omega)}^{=0} = X(f)\cdot\omega$$

Cartan

$$\| $$

$$d(i_X(f\omega)) + i_X(\overbrace{d(f\omega)}^{=0}) = d(i_X(f\cdot\omega))$$

Si supp(f) compact $\displaystyle\int_M X(f)\omega = \int_{U\supseteq \mathrm{supp}(f)} d(i_X(f\cdot\omega)) \overset{\text{Stokes}}{=} \int_{\partial U} i_X(f\cdot\omega) = 0$

Si non $\displaystyle 0 = \lim_{n\to+\infty}\int_M X(\sigma_n\cdot f)\omega = \lim_{n\to+\infty}\int_M \sigma_n\cdot X(f)\omega + \lim_{n\to+\infty}\int_M X(\sigma_n)f\omega = (+)$

$$(+) = \int_M X(f)\cdot\omega + \underbrace{\lim_{r\to+\infty} \int_M X(\sigma_r)\, f \cdot \omega} = 0$$

$$\underbrace{\left| \int_M X(\sigma_r)\, f\, \omega \right|}_{\wedge|} \le \int_M \underbrace{\left|X(\sigma_r)(x)\right|_x} \cdot |f|\, \omega$$

Cauchy-Schwartz

$$= \langle (\nabla\sigma_r)(x), X(x)\rangle_x \le \underbrace{\left|(\nabla\sigma_r)(x)\right|_x}_{\le C/r} \cdot \overbrace{\left|X(x)\right|_x}^{\le B} \le \frac{C'}{r}$$

$$\frac{C'}{r} \cdot \underbrace{\int_M |f| \cdot\omega}_{<+\infty} \xrightarrow{r\to+\infty} 0$$

$\square$

__Cor__ $X : M \to TM$, $\mathcal{L}_X(\omega) = 0$, $\max_{x \in M} |X(x)|_x < +\infty$; $f, g \in C^1(M, E)$, E euclidien,

$$\underbrace{h(x) := \langle f(x), g(x) \rangle_E, \; x \mapsto \langle X(f)(x), g(x) \rangle_E, \; x \mapsto \langle f(x), X(g)(x) \rangle_E}_{\in L^1(M,E)}$$

C'est valable si $f, g, X(f), X(g) \in C^2(M, E)$

$$\Rightarrow \langle Xf, g \rangle_M + \langle f, (Xg) \rangle_M = 0$$

__Preuve__ $X(h)(x) = \langle (Xf)(x), g(x) \rangle_E + \langle f(x), X(g)(x) \rangle_E$. Applique la prop. à h.

2.2

(3) **Prop.** M var. rien. complete, orientée, $\alpha \in \Omega^k(M)$, $\beta \in \Omega^{k+1}(M)$ t.q.

$$\underbrace{x \mapsto |\alpha(x)|_x \cdot |\beta(x)|_x,\ x \mapsto \langle (d\alpha)(x), \beta(x) \rangle_x,\ x \mapsto \langle \alpha(x), \delta(\beta)(x) \rangle_x}_{\in L^1(M, \mathbb{R})}$$

$$\Rightarrow \langle d\alpha, \beta \rangle_M = \langle \alpha, \delta\beta \rangle_M$$

Dém $0 \overset{?}{=} \langle d\alpha, \beta \rangle_M - \langle \alpha, \delta\beta \rangle_M = \int_M d(\alpha \wedge (*\beta)) - \alpha \wedge (*(\delta\beta))$

$= d\alpha \wedge (*\beta) + (-1)^k \cdot \alpha \wedge d(*\beta)$, $\quad d(*\beta) = (-1)^{k+1} \cdot *(\delta\beta)$

$$\left[\langle \eta, \xi \rangle_M = \int_M \eta \wedge (*\xi),\quad \delta := (-1)^{d(k+1)+1} (* \circ d \circ *) \right]$$

Si $\operatorname{supp}(\alpha)$ ou $\operatorname{supp}(\beta)$ sont compactes

$$\langle d\alpha,\beta\rangle_M-\langle\alpha,\delta\beta\rangle_M=\int_M d(\alpha\wedge(*\beta))=\int_U d(\alpha\wedge(*\beta))\overset{\text{Stokes}}{=}\int_{\partial U}\alpha\wedge(*\beta)=0$$

Si non $\langle\alpha,\delta\beta\rangle_M=\lim_{r\to+\infty}\langle\sigma_r\alpha,\delta\beta\rangle_M=\lim_{r\to+\infty}\langle d(\sigma_r\alpha),\beta\rangle_M=$

$$=\lim_{r\to+\infty}\langle\sigma_r\,d\alpha,\beta\rangle_M+\lim_{r\to+\infty}\langle d(\sigma_r)\wedge\alpha,\beta\rangle_M=$$

$$=\langle d\alpha,\beta\rangle_M+\lim_{r\to+\infty}\langle d(\sigma_r)\wedge\alpha,\beta\rangle_M,\text{ mais}$$

$$\langle d(\sigma_r)\wedge\alpha,\beta\rangle_M\leq\|d\sigma_r\|_M\cdot\|\alpha\|_M\cdot\|\beta\|_M\leq\frac{c}{r}\cdot\overbrace{\int_M|\alpha(x)|_x\cdot|\beta(x)|_x\,\omega}^{<+\infty}\xrightarrow[r\to+\infty]{}0$$

Cor. $\alpha, \delta(\beta) \in \Omega^k_{(2)}(M)$, $d\alpha, \beta \in \Omega^k_{(2)}(M) \Rightarrow \langle \alpha, \delta\beta \rangle_M = \langle d\alpha, \beta \rangle_M$.

④ **Thm** (Andreotti-Vesentini) $\mathcal{H}^k_{(2)}(M) = \Omega^k_{(2)}(M)^{d=\delta=0}$

Dem "$\supseteq$": Evident, $\Delta = d\circ\delta + \delta\circ d$.

"$\subseteq$": $\alpha \in \mathcal{H}^k_{(2)}(M)$

$$\begin{pmatrix} d\alpha \overset{?}{=} 0 \\ \delta(\alpha) \overset{?}{=} 0 \end{pmatrix} \Leftrightarrow \left(\underbrace{|d\alpha|^2_H + |\delta\alpha|^2_H}_{"} \overset{?}{=} 0 \right) \Leftrightarrow$$

$$\Leftrightarrow \lim_{r \to +\infty} \underbrace{|\sigma_r \cdot d(\alpha)|^2_H}_{} + \underbrace{|\sigma_r \cdot \delta(\alpha)|^2_H}_{} \overset{?}{=} 0$$

$$|\sigma_r d(\alpha)|^2_H = \langle \sigma_r d\alpha, \sigma_r d\alpha\rangle_H = \langle d(\alpha), \underbrace{\sigma_r^2 \cdot d\alpha}_{= d(\sigma_r^2\alpha) - 2\sigma_r\cdot d(\sigma_r)\wedge\alpha}\rangle_H =$$

$$= \overbrace{\langle \delta(d\alpha), \sigma_r^2\alpha\rangle_H} - \langle \sigma_r\cdot d(\alpha), 2\, d(\sigma_r)\wedge\alpha\rangle_H$$

$$|\sigma_r\cdot\delta(\alpha)|^2_H = \langle \delta(\alpha), \underbrace{\sigma_r^2\cdot\delta(\alpha)}_{= \delta(\sigma_r^2\cdot\alpha) \pm 2\sigma_r * (d(\sigma_r)\wedge(*\alpha))}\rangle_H = \overbrace{\langle d(\delta(\alpha)), \sigma_r^2\alpha\rangle} \pm \langle \sigma_r\cdot\delta(\alpha), 2 * (d(\sigma_r)\wedge(*\alpha))\rangle$$

$$|\sigma_r\, d(\alpha)|^2_H + |\sigma_r\cdot\delta(\alpha)|^2_H = \overbrace{\langle \Delta(\alpha), \sigma_r^2\alpha\rangle_H}^{=0} + \cdots \pm \cdots \xrightarrow[r\to+\infty]{} 0$$

$$\langle \eta, \xi \rangle_M \le |\eta|_M \cdot |\xi|_M = \sqrt{|\eta|_M^2 \cdot |\xi|_M^2} \le \frac{|\eta|_M^2 + |\xi|_M^2}{2}$$

(5) **Prop. 2.5** $\mathcal{H}^k_{(2)}(M) \cap d\left(\Omega^{k-1}_{(2)}(M)\right) = 0$

Dem. $\alpha \in \Omega^{k-1}_{(2)}(M)$, $d\alpha \in \mathcal{H}^k_{(2)}(M)$, $\left(d\alpha \overset{?}{=} 0\right) \iff \left(|d\alpha|^2_M \overset{?}{=} 0\right)$.

$$|d\alpha|^2_M = \langle d\alpha, d\alpha \rangle_M = \langle \overbrace{\delta(d(\alpha))}^{=0}, \alpha \rangle_M = 0$$

Andreotti-Vesentini

(6) G : groupe de Lie réel, semi simple, $K \leq G$ compact maximal,
$([G:G^0]<+\infty)$ $X := K \backslash G$

Ex. $G = \tilde{G}(\mathbb{R})$, $\tilde{G} := \mathrm{Res}_{F/\mathbb{Q}}(SL_n)$

$\begin{pmatrix} G \\ \cup \\ K \end{pmatrix} \leadsto \begin{pmatrix} \mathfrak{g} \\ \cup \\ \mathfrak{k} \end{pmatrix}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p} := (\mathfrak{k})^\perp$ par la forme de Killing,
$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $B(x,y) := \mathrm{tr}(\underbrace{[x,[y,\cdot]]}_{= \mathrm{ad}(x) \circ \mathrm{ad}(y)})$

$\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta|_{\mathfrak{k}} = \mathrm{Id}_{\mathfrak{k}}$, $\theta|_{\mathfrak{p}} = -\mathrm{Id}_{\mathfrak{p}}$,

$B_\theta(x,y) := -B(x,\theta(y))$ défini positif. Avec $X \curvearrowleft G$, nous avons une structure riemannienne sur X induite par B_θ.